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Theoretical analysis of heat transfer in laminar pulsating flow

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Abstract

Pulsation effect on heat transfer in laminar incompressible flow, which led to contradictory results in previous studies, is theoretically investigated in this work starting from basic principles in an attempt to eliminate existing confusion at various levels. First, the analytical solution of the fully developed thermal and hydraulic profiles under constant wall heat flux is obtained. It eliminates the confusion resulting from a previously published erroneous solution. The physical implications of the solution are discussed. Also, a new time average heat transfer coefficient for pulsating flow is carefully defined such as to produce results that are both useful from the engineering point of view, and compliant with the energy balance. This rationally derived average is compared with intuitive averages used in the literature. New results are numerically obtained for the thermally developing region with a fully developed velocity profile. Different types of thermal boundary conditions are considered, including the effect of wall thermal inertia. The effects of Reynold and Prandtl numbers, as well as pulsation amplitude and frequency on heat transfer are investigated. The mechanism by which pulsation affects the developing region, by creating damped oscillations along the tube length of the time average Nusselt number, is explained.

Keywords: Forced convection; Pulsating flow; Laminar; Incompressible; Thermally developing

1. Introduction

Pulsatile flow is frequently encountered in natural systems (circulatory system, respiratory system) as well as engineering systems (reciprocating pumps, IC engines, pulse combustors). It manifests itself in internal as well as external flow situations (hot wire anemometer in a fluctuating flow, tube bundles where vortex shedding from the leading tube induces flow fluctuations on subsequent tubes). Flow fluctuations may also be intentionally imposed in an attempt to improve heat transfer (pulsating jet cooling). Although the problem has received considerable attention, available results are contradictory, and the question is still open: Does pulsation enhance or else degrade heat transfer compared to steady flow?

In earlier studies, heat and mass transfer in pulsating flow with zero net flow have received considerable attention [1–3]. Fluid pulsates in a duct connecting two reservoirs maintained at uniform temperature (or concentration). Both theoretically and experimentally a tremendous improvement was noticed compared to the non-pulsating case, which corresponds to pure conduction through the fluid between the two reservoirs. This result cannot be extrapolated to the case of non-zero net flow, since the basis of comparison is not the same (conduction in one case and steady convection in the other). For the sake of simplicity, the analysis will be restricted to the case where superimposed pulsation amplitude is less than the average flow. Hence, flow reversal is precluded, which simplifies boundary conditions on both ends.

Kim et al. [4] numerically studied heat transfer in a channel subject to a pulsating flow that enters the duct with constant temperature. The study was made for the thermally developing region of the channel, with a fully developed velocity profile. A constant surface temperature was imposed at channel walls. The well-known SIMPLER algorithm [5] based on the finite difference method was used to solve the problem. They showed

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Nomenclature

A_o	average pressure gradient (N/m^3)
A_1	amplitude of pressure gradient oscillations (N/m^3)
A_c	cross-sectional area (m^2)
$\mathbf{a}_x$, $\mathbf{a}_y$	unit vectors along the flow and normal to the flow, respectively
Bi	Biot number, $h_{out}R/k_f$
C	heat capacity (J/kg K)
D	channel half depth (m)
f	dependence of θ_1 on r (Eq. (3.4))
G	Green's function of Eq. (3.6)
g	amplitude of oscillating velocity (Eq. (2.11b))
h	heat transfer coefficient (W/m^2 K)
Ir	inertia ratio, $(s\rho_s C_s)/(R\rho_f C_f)$
i	imaginary factor
J_0	Bessel function of the first kind of order 0
K	constant defined by (3.5b)
k	thermal conductivity (W/m K)
L	channel length (m)
$\mathbf{n}$	unit outward normal
Nu	Nusselt number, hR/k
P	pressure (N/m^2)
P	duct perimeter (m)
Pr	Prandtl number, ν/α
q	heat flux (W/m^2)
R	duct radius (m)
r	radial coordinate
Re	Reynold number, $u_c R/\nu$
s	duct wall thickness (m)
T	temperature (K)
t	time
$\mathbf{V}$	velocity vector (m/s)
V	volume (m^3)
u, v	velocity components
u_c	time average centerline velocity (m/s)
x, y	coordinates along the flow and normal to it, respectively
Y_0	Bessel function of the second kind of order 0

Greek symbols

α	thermal diffusivity (m^2/s)
β	dimensionless amplitude of pressure gradient oscillations
δ	Dirac distribution
θ	dimensionless temperature
ν	kinematic viscosity (m^2/s)
ρ	density (kg/m^3)
τ	dimensionless period of oscillation, $\tau'\nu/R^2$
Ω, ω	dimensionless and dimensional frequency

Subscripts

amb	ambient
b	bulk
f	fluid
in	inlet
o	time average quantity
out	outside
ref	reference
s	solid
st	steady problem
t	time dependent quantity
w	wall

Superscripts

$'$	dimensional variable (used for r, x, y, t, u, v and τ)

that in the fully developed region the difference between the time average heat flux and the heat flux corresponding to the steady flow case was generally small. One may argue that the type of boundary conditions used was too restrictive. Wall temperature was assumed constant, both in space and time. This ties the heat exchange process, and may reduce the effect of pulsation.

Moschandreou and Zamir [6] analytically studied pulsating flow in the thermally fully developed region in a circular tube with an imposed constant wall heat flux. They have found both positive and negative differences (depending on operating conditions) between the time average Nusselt number for pulsating flow Nu_o, and the corresponding Nusselt number for steady flow Nu_{st}. However, their solution suggests the following strange result: for very high frequency the difference between Nu_o and Nu_{st} continues to grow. As is well known, any inertial system has a cut-off frequency above which it should not be sensitive to oscillating excitations. In fact, their analytical solution for the fully developed case was erroneous and will be redone here, in order to obtain the correct solution.

Compressibility may play a role in pulsating flow [7] for tubes with constant wall heat flux using the method of characteristics and assuming a flat velocity profile. Pulsation effects were important when the frequency was near that of resonance (i.e. sound speed divided by tube length). In this work, pulsation frequency will be assumed much lower than resonant frequency, which applies to most engineering applications. Hence the analysis will be restricted to the incompressible case. On the other hand velocity and temperature profiles will not be considered as flat.

In a very interesting study [8], it was shown that for the same spatial and temporal temperature distribution, different definitions of average Nusselt number Nu_o for pulsating flow will lead to contradictory results. This means that we may observe either enhancement or degradation in Nu_o, depending on how time average was constructed from the same raw data. They have intuitively constructed four different definitions for the constant wall heat flux case. They have recommended the use of one of them, without giving a rigorous theoretical basis for that choice. A rational approach will be proposed here in order to construct a new definition of Nu_o based on the energy equation and Newton's law of cooling, which can in addition be applied to other cases of boundary conditions as well as in the thermally developing region.

Turbulent pulsating external flow has also received attention in recent publications [9,10]. For turbulent flow, it seems to be established that pulsation enhances heat transfer. The effect is even higher if pulsation was able to induce turbulence in an otherwise laminar flow. Effects of pulsation on laminar–turbulent transitions were also investigated [11]. This work will be devoted to laminar flow, for which published results were not conclusive.

The thermally developing region did not receive sufficient attention in the literature. It is to be expected that this region should be more sensitive to pulsation, and that it should extend to a longer distance than that of the corresponding steady flow. This was confirmed by the numerical study of Kim et al. [4] for flow in 2D channel with constant wall temperature. They found that the difference between the time average Nusselt number, according to their definition of that average, and Nu_{st} fluctuated along the channel length. The difference was greater than the fully developed case, though still relatively small. It started with a negative value in the first part of the developing region. Since oscillations were damped, the average along the channel was rather negative, i.e. pulsation degraded heat transfer. Wall temperature was assumed constant both in space and time. It is to be expected that other less restrictive boundary conditions would manifest greater sensitivity to pulsation. Hafez and Montasser [7] have found that the difference between pulsating and non-pulsating flow decreases along the tube length, which confirms the fact that the thermally developing region is more sensitive to pulsation.

To the authors knowledge, the effect of non-ideal boundary conditions had never been addressed neither experimentally nor theoretically. Hence, in this work, the developing region will be systematically studied in order to investigate the effect of different parameters, under different types of boundary conditions. Two different cases of non-ideal boundary conditions will be investigated. The first case is that of a finite thermal resistance between fluid and its environment, also called Robin type boundary condition. In the second case, tube wall will be considered to have a finite thermal capacity. Moreover, some discussions will be given about the expected effect of non-linear boundary conditions (natural convection, radiation, ...).

In the sequel, model equations will be derived, together with an adequate set of boundary conditions, cast in a non-dimensional form. Section 3 will present the analytical solution for the fully developed constant wall heat flux case. Previous definitions of average Nu will be critically revised in Section 4, to derive a new general definition. The thermally developing problem will be considered in Sections 5–7 for different kinds of non-ideal boundary conditions. This will be followed by conclusions in Section 8.

2. Assumptions and governing equations

As a result of the above analysis pulsating flow will be considered here, under the following simplifying assumptions:

1. The flow is assumed laminar and incompressible.
2. Pulsation amplitude does not allow flow reversal.
3. The velocity profile is fully developed but the thermal boundary layer is developing. This applies to fluids with high Prandtl number Pr, or moderate Pr with an unheated section in the upstream side. It includes most common engineering applications. It precludes the case of liquid metals where the thermal boundary layer development is not an issue.
4. Second-order effects, like the variation of thermophysical properties and viscous dissipation, are neglected for simplicity.

Under these conditions, mass, momentum and energy equations are written as

$$\nabla \cdot \mathbf{V} = 0, \tag{2.1a}$$

$$\partial \mathbf{V}/\partial t' = -\nabla P/\rho_f + \nu_f \nabla^2 \mathbf{V}, \tag{2.1b}$$

$$\partial T/\partial t' + \mathbf{V} \cdot \nabla T = \alpha_f \nabla^2 T, \tag{2.1c}$$

where $\mathbf{V}$ is the velocity vector, t' the time, P the pressure, T the temperature, ρ_f, ν_f and α_f are, respectively, the fluid density, kinematic viscosity and thermal diffusivity. Since the velocity profile is assumed fully developed, the hydrodynamic part would be completely specified by imposing a pressure gradient, and boundary conditions on the axis and at the wall:

$$\nabla P = -\mathbf{a}_x \left(A_o + A_1 \cos \omega t'\right), \tag{2.2}$$

$$\partial u'/\partial r' = v' = 0 \quad \text{for } r' = 0, \tag{2.3a}$$

$$u' = v' = 0 \quad \text{for } r' = R. \tag{2.3b}$$

The pressure gradient contains a steady and a pulsating part, of amplitudes A_o and A_1, respectively. The unit vector $\mathbf{a}_x$ is in the x-direction parallel to the flow (Fig. 1), ω is the frequency, r' is the radial coordinate (normal to

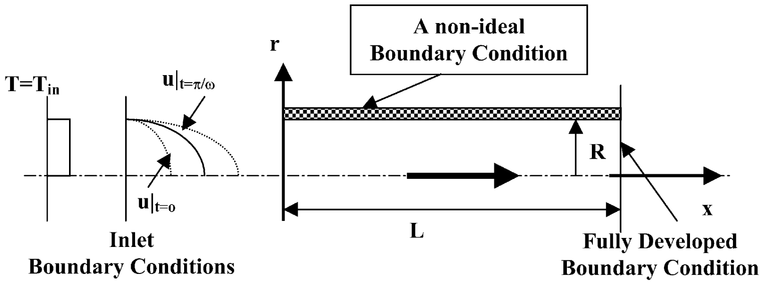


Fig. 1. Problem description.

flow direction), R is the tube radius, and u' and v' are the axial and radial velocity components. Boundary conditions should be imposed on temperature all around the domain, since the energy equation is elliptic in space. Constant temperature is assumed at inlet, and fully developed temperature profile at outlet:

$$\text{for } x' = 0: \quad T = T_{\text{in}}, \tag{2.4a}$$

$$\text{for } x' = L: \quad \partial/\partial x'[(T_{\text{w}} - T)/(T_{\text{w}} - T_{\text{b}})] = 0, \tag{2.4b}$$

where T_{w} is the wall temperature and T_{b} is the bulk temperature defined as

$$T_{\text{b}} = \int Tu' \,\mathrm{d}a \bigg/ \int u' \,\mathrm{d}a \tag{2.4c}$$

in which da is the element of area normal to flow. At the centerline we have from symmetry,

$$\text{for } r' = 0: \quad \partial T/\partial r' = 0. \tag{2.5}$$

Finally, at the wall, the more realistic case of a finite wall thermal resistance will be considered (the wall thermal capacitance will be considered in Section 7):

$$k_{\text{f}}\,\partial T/\partial r'|_{r=R} = h_{\text{out}}(T_{\text{amb}} - T|_{r=R}) + q_{\text{w}}, \tag{2.6}$$

where k_{f} is the fluid thermal conductivity, h_{out} is an equivalent coefficient related to external resistances, T_{amb} is the ambient temperature and q_{w} is an external source of added heat.

In order to transform the above system into a non-dimensional form, use will be made of the following transformations:

$$\begin{aligned} &x = x'/(R\,Re), \quad r = r'/R, \quad t = t'\nu_{\text{f}}/R^2, \\ &u = u'/u_{\text{c}}, \quad \theta = (T - T_{\text{in}})/\Delta T_{\text{ref}}, \end{aligned} \tag{2.7}$$

where $Re = u_{\text{c}}R/\nu$ is the Reynolds number and u_{c} is the time average of the centerline velocity. The reference temperature difference ΔT_{ref} in this case should be valid in both extremes (constant temperature, constant heat flux) as well as in all intermediate cases:

$$\Delta T_{\text{ref}} = \frac{Bi}{1+Bi}(T_{\text{amb}} - T_{\text{in}}) + \frac{1}{1+Bi}\frac{q_{\text{w}}R}{k_{\text{f}}}, \tag{2.8}$$

where $Bi = h_{\text{out}}R/k_{\text{f}}$ is the Biot number. The constant wall temperature corresponds to $Bi \to \infty$, while the constant heat flux corresponds to $Bi = 0$.

Using the problem linearity, Womersley [12] has analytically obtained

$$u(r,t) = u'/u_{\text{c}} = u_{\text{o}}(r) + \beta u_1(r,t), \tag{2.9}$$

where

$$u_{\text{o}}(r) = 1 - r^2 = u_{\text{st}}(r), \tag{2.10}$$

$$u_1(r,t) = g(r)\mathrm{e}^{\mathrm{i}\Omega t}, \tag{2.11a}$$

$$g(r) = (-\mathrm{i}/\Omega)\lfloor 1 - J_0(\sqrt{-\mathrm{i}\Omega}r)/J_0(\sqrt{-\mathrm{i}\Omega})\rfloor, \tag{2.11b}$$

$\beta = 4A_1/A_{\text{o}}$; $\Omega = \omega R^2/\nu_{\text{f}}$; and the reference velocity is $u_{\text{c}} = -A_{\text{o}}R^2/4\rho_{\text{f}}\nu_{\text{f}}$.

It remains to solve the dimensionless energy equation, which takes the form

$$Pr(\partial\theta/\partial t + u\,\partial\theta/\partial x) = \nabla^{2^*}\theta, \tag{2.12}$$

where the RHS operator is a modified Laplacian operator

$$\nabla^{2^*}\theta = (1/Re^2)\partial^2\theta/\partial x^2 + \partial^2\theta/\partial r^2 + 1/r\,\partial\theta/\partial r. \tag{2.13}$$

Problem control parameters are thus: Ω, β, Pr, Re and Bi. Note that scaling x with Re, confines the effect of Re into a single term which is axial conduction. This term vanishes in the fully developed region and is expected to be small in the developing region. Hence the role of Re is limited with this scaling of x. A fact that is of course only valid for laminar flow. Values of β will be limited such as to avoid flow reversal. No initial conditions will be imposed, since we are interested in the periodic steady state.

3. Fully developed flow at constant wall heat flux

In order to obtain an analytical solution of the fully developed temperature profile θ and constant wall heat flux Moschandreou and Zamir [6] have split θ into two parts

$$\theta(x,r,t) = \theta_{\text{o}}(x,r) + \beta\theta_1(r,t), \tag{3.1}$$

where θ_o represents the steady part, which depends linearly on x due to the constant wall heat flux condition ($\partial\theta_o/\partial x = 4/Pr$), while θ_1 represents the pure oscillating part. Substituting (3.1) in (2.12) yields two differential systems, one for the time average part:

$$Pr\, u_o\, \partial\theta_o/\partial x = \nabla^{2^*}\theta_o, \tag{3.2a}$$

$$\theta_o(0,r) = 0, \quad \partial\theta_o/\partial r|_{r=1} = 1, \quad \partial\theta_o/\partial r|_{r=0} = 0 \tag{3.2b}$$

and the other for the transient part:

$$Pr(\partial\theta_1/\partial t + u_1\, \partial\theta_0/\partial x) = \nabla^{2^*}\theta_1, \tag{3.3a}$$

$$\partial\theta_1/\partial r|_{r=1} = 0, \quad \partial\theta_1/\partial r|_{r=0} = 0. \tag{3.3b}$$

The solution of (3.2a) and (3.2b) is the same as that of the fully developed steady problem $\theta_o = \theta_{st}$ [13]. Moschandreou and Zamir [6] have seeked θ_1 in the form

$$\theta_1(r,t) = 4f(r)\,e^{i\Omega t}. \tag{3.4}$$

Substituting this into Eq. (3.3a) and (3.3b) leads to the ordinary differential equation

$$(rf')' + K^2 rf = r g(r), \tag{3.5a}$$

where

$$K^2 = -i\Omega Pr \tag{3.5b}$$

with the following boundary conditions

$$f'(0) = f'(1) = 0. \tag{3.5c}$$

Their analytical solution of (3.5a)–(3.5c) was not correct (Appendix A). The correct solution will be obtained here using Green's function $G(r, r'')$ satisfying

$$\begin{aligned} &\partial/\partial r(r\,\partial/\partial r G(r,r'')) + K^2 r G(r,r'') = \delta(r - r''),\\ &\quad \partial G/\partial r|_{r=0;r=1} = 0, \end{aligned} \tag{3.6}$$

where $\delta(r - r'')$ is the Dirac distribution. The solution has the general form

$$G(r,r'') = \begin{cases} C_{11}J_0(Kr) + C_{12}Y_0(Kr), & 0 \leqslant r < r'', \\ C_{21}J_0(Kr) + C_{22}Y_0(Kr), & r'' < r \leqslant 1, \end{cases} \tag{3.7}$$

where J_0 and Y_0 are the Bessel functions of the first and second kinds, respectively, of order 0 and C_{ij} are complex functions of r'' (i.e., constants with respect to r). To determine them, we need to use boundary conditions at $r = 0, 1$ as well as jump conditions at $r = r''$:

$$G|_{r=r''-0}^{r=r''+0} = 0, \quad r\,\partial G/\partial r|_{r=r''-0}^{r=r''+0} = 1. \tag{3.8}$$

Hence

$$\begin{aligned} C_{11} &= \pi\big[Y_0(Kr'')J_0'(K) \\ &\qquad - Y_0'(K)J_0(Kr'')\big]/2J_0'(K), \\ C_{12} &= 0, \end{aligned} \tag{3.9a}$$

$$\begin{aligned} C_{21} &= -\pi J_0(Kr'')Y_0'(K)/2J_0'(K), \\ C_{22} &= \pi J_0(Kr'')/2. \end{aligned} \tag{3.9b}$$

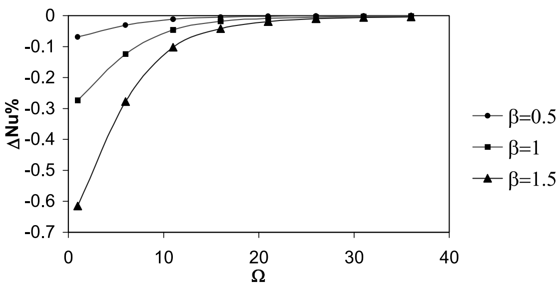


Fig. 2. Effect of pulsation frequency Ω and amplitude β on Nu.

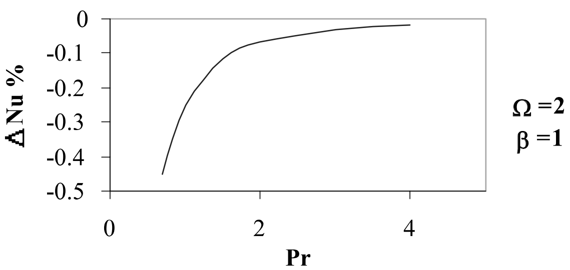


Fig. 3. Effect of Pr on Nu.

Substituting of the constants from (3.9a) and (3.9b) into (3.7) we get G. Hence f is written as

$$f(r) = \int_0^1 G(r'',r)r''g(r'')\,dr''. \tag{3.10}$$

The erroneous solution [6] has resulted in an unphysical behavior in their results where the difference $\Delta Nu = Nu_o - Nu_{st}$ continuously increases for large Ω. This is not realistic since any inertial system should have a cut-off frequency beyond which it should not respond to external excitations. Their solution showed a maximum point at which heat transfer would be highly enhanced by pulsation, which resulted in a great confusion in the literature. Using the corrected temperature profile obtained here with their definition of the average Nusselt number ((4.4) below), we find that (Fig. 2):

1. the difference ΔNu is always negative, neither maxima nor minima were observed;
2. the difference ΔNu is very small, less than 1% of the steady Nu;
3. the difference ΔNu tends to zero as Ω tends to infinity.

As expected pulsation effect decreases as Pr increases (Fig. 3).

4. The definition of average Nusselt number

The definition of average Nusselt number is not a trivial matter [8]. Kim et al. [4] studied the case of flow in a channel of depth 2D subject to constant wall temperature T_w. They defined the instantaneous Nusselt number and its average as

$$Nu = D\partial[(T - T_{\text{inlet}})/(T_w - T_{\text{inlet}})]/\partial y'\big|_{y'=D}, \tag{4.1}$$

$$Nu_o = 1/\tau \int_0^\tau Nu \, dt, \tag{4.2}$$

where y' is the distance from the center line normal to flow direction and τ is the dimensionless period of oscillation. Expression (4.1) can be rewritten as

$$Nu = [q_w/(T_w - T_{\text{inlet}})]D/k. \tag{4.1$'$}$$

The ratio between square brackets does *not* represent the local instantaneous heat transfer coefficient. The latter requires that the temperature difference be between wall and fluid bulk, not inlet. The above definition was most probably chosen in order to avoid a problem that will be solved in this section. In fact, if both numerator and denominator in (4.1′) were time varying, then the time average Nusselt number (4.2) would have required a more elaborate analysis to define, which was not the object of their work. For the constant wall heat flux case the following definition was proposed [6]:

$$Nu = (q_w/T_w - T_b)R/k = 2/(\theta_w - \theta_b) \tag{4.3}$$

to obtain the average Nu_o in the form

$$Nu_o = 2/(\theta_{wo} - \theta_{bo}), \tag{4.4}$$

where

$$\theta_{wo} = 1/\tau \int_0^\tau \theta_w \, dt. \tag{4.5}$$

As for the average bulk temperature, two expressions were given, the second was used:

$$\theta_{bo} = \int_0^\tau \left(\int \theta u \, da \right) dt \Big/ \int_0^\tau \left(\int u \, da \right) dt, \tag{4.6a}$$

$$\theta_{bo} - \theta_{st\,b} = \int_0^\tau \int (\theta - \theta_{st})(u - u_{st}) \, da \, dt \Big/ \int_0^\tau \int u \, da \, dt. \tag{4.6b}$$

In comparing Nu_o with $Nu_{st} = 2/(\theta_{st\,w} - \theta_{st\,b})$ they have implicitly assumed that $\theta_{w\,o}$ is the same as $\theta_{st\,w}$, which is strictly valid for the fully developed temperature field only. In fact, if we take the time average of the energy equation, we get

$$Pr 1/\tau \int_0^\tau u \, \partial\theta/\partial x \, dt = \nabla^{2^*} \theta_o. \tag{4.7}$$

Comparing it with the steady equation

$$Pr \, u_{st} \, \partial\theta_{st}/\partial x = \nabla^{2^*} \theta_{st}, \tag{4.8}$$

we find that both equations are equivalent if and only if

$$1/\tau \int_0^\tau u \, \partial\theta/\partial x \, dt = \left(1/\tau \int_0^\tau u \, dt \right) \left(1/\tau \int_0^\tau \partial\theta/\partial x \, dt \right) = u_o \, \partial\theta_o/\partial x. \tag{4.9}$$

This last condition is satisfied for fully developed temperature profile only, where the axial temperature gradient becomes constant independent of space and time. Hence, the definition of the time average Nusselt number is also limited to the case studied, i.e. fully developed constant wall heat flux.

Guo and Sung [8] studied the definition of time average Nu for the constant wall heat flux case. They have proposed four different definitions summarized in Table 1.

Values of $Nu_o^{(1)}$ were always less than Nu_{st}. Values of $Nu_o^{(2)}$ are still less than Nu_{st}, except for low Ω and very high β (near flow reversal) where it may slightly exceed Nu_{st}. It is clear that $Nu_o^{(3)}$ is an arithmetic mean value of the instantaneous Nu, whereas $Nu_o^{(2)}$ is a harmonic mean one. Obviously, $Nu_o^{(3)} > Nu_o^{(2)}$. Values of $Nu_o^{(3)}$ are higher

Table 1
Definitions of average Nu proposed by Guo and Sung [8]

1	2	3	4
$Nu_o^{(1,2)} = 2/(\theta_{wo} - \theta_{bo})$		$Nu_o^{(3,4)} = 1/\tau\int_0^\tau 2/(\theta_w - \theta_{bt})dt$	
$\theta_{bo} = \dfrac{\int_0^\tau\left(\int\theta u da\right)dt}{\int_0^\tau\left(\int u da\right)dt}$	$\theta_{bo} = 1/\tau\int_0^\tau\theta_{bt}dt$ $\theta_{bt} = \int\theta u\,da\Big/\int u\,da$	$\theta_{bt} = \dfrac{\int\theta u\,da}{\int u\,da}$	$\theta_{bt} = \dfrac{\int\theta\lvert u\rvert da}{\int\lvert u\rvert da}$
$\theta_{w0} = \theta_{st\,w} +$ $\int_0^\tau\int(\theta - \theta_{st})(u - u_{st})da\,dt\Big/\int_0^\tau\int u\,da\,dt$	$\theta_{wo} = 1/\tau\int_0^\tau\theta_w dt$		

than Nu_{st} for low frequencies, even with moderate amplitudes. Definition 4 was constructed to deal with the case of flow reversal. Guo and Sung [8] had the merit of pointing out the importance of the definition of Nu as well as proposing a definition valid for the flow reversal case. However, no argument was given to select the appropriate definition. Moreover, all definitions concerned the constant wall heat flux case.

In this work, the time average Nu will be based on the time average heat transfer coefficient h_o, which will be defined here in a rational way satisfying the following criteria:

1. It should be usable in Newton's law of cooling $q = h_o * \text{area} * (T_{wo} - T_{bo})$ in which T_{wo} and T_{bo} are the time averages of wall and bulk temperatures, to give a heat flux q that is compatible with the energy balance.
2. The thermal resistance $1/(h_o * \text{area})$ should be usable in series or parallel with other thermal resistances in the system, to get the correct overall heat transfer coefficient.

In order to define T_{bo} let us integrate the energy equation over a slice of width dx':

$$\partial/\partial t' \oint_V T \, dV + \oint_{A_c} \mathbf{n} \cdot \mathbf{V} T \, da \bigg|_{x'}^{x'+dx'} = \alpha_f \left[\oint_{A_c} \mathbf{a}_x \cdot \nabla T \, da \bigg|_{x'}^{x'+dx'} + \mathsf{P} \, dx' \mathbf{n}.\nabla T|_w \right], \tag{4.10}$$

where P is the circumference of the duct, A_c its cross-sectional area, V the volume and $\mathbf{n}$ is the unit outward normal. The first term in the RHS represents the difference between axial conduction on both sides of the slice and can be safely neglected. The first term in the LHS represents the fluid thermal inertia. It vanishes by taking the time-average of (4.10):

$$\left(\rho_f C_f/\tau'\right) \int_0^{\tau'} \left(\oint_{A_c} \mathbf{n} \cdot \mathbf{V} T \, da \bigg|_{x'}^{x'+dx'} \right) dt' = \left(\mathsf{P} \, dx'/\tau'\right) \int_0^{\tau'} k_f \, \mathbf{n} \cdot \nabla T \, dt' = \mathsf{P} \, dx' q_{wo}, \tag{4.11}$$

where C_f is the heat capacity. Hence in order to be able to write a simple expression like

$$\dot{m}_o C_f \, dT_{bo} = \mathsf{P} \, dx' q_{wo}, \tag{4.12}$$

the definition of T_{bo} in pulsating flow must be in the form

$$T_{bo} = \left(1/\tau'\right) \int_0^{\tau'} \left(\oint_{A_c} \rho_f \mathbf{n} \cdot \mathbf{V} T \, da \right) dt' \bigg/ \dot{m}_o, \tag{4.13a}$$

where

$$\dot{m}_o = \left(1/\tau'\right) \int_0^{\tau'} \left(\oint_{A_c} \rho_f \mathbf{n} \cdot \mathbf{V} \, da \right) dt'. \tag{4.13b}$$

Hence, definition (4.6a), which appears in $Nu^{(1)}$, had a physical meaning, but all other definitions appearing in $Nu^{(2,3,4)}$ were groundless. To define the average wall temperature, suppose an external resistance was connected in series or parallel that is characterized by a coefficient h_{out}:

$$q_w = h_{out}(T_{amb} - T_w), \tag{4.14}$$

where T_{amb} is the ambient air temperature. Since both the resistance and the ambient temperature are steady, the heat exchanged in a cycle, should be function of the time average wall temperature T_{wo} defined as

$$T_{wo} = \left(1/\tau'\right) \int_0^{\tau'} T_w \, dt'. \tag{4.15}$$

Using (4.15) guarantees the satisfaction of criterion B, which is not the case of $Nu^{(1)}$. Hence, none of the definitions proposed by Guo and Sung [8] could satisfy both criteria A and B. The final expression of h_o is

$$h_o = \left(k_f/\tau'\right) \int_0^{\tau'} \mathbf{n} \cdot \nabla T|_w \, dt' / (T_{wo} - T_{bo}), \tag{4.16}$$

where T_{wo} and T_{bo} are given by (4.15) and (4.13a), (4.13b). The definition of time average h given here is thus rationally built such as to guarantee that whenever the overall rate of heat transfer is enhanced (or degraded) in pulsating flow, then h_o will increase (or decrease).

5. Numerical solution in the thermally developing region

Having established the fact that pulsation had negligible effect on the thermally developed region, it remains to examine the thermally developing case. The latter should be relatively more sensible to pulsation, but the quantification of this effect could only be made through a numerical solution. This section is devoted to a presentation of the code used, which is based on the finite element method (FEM) including all verifications done to guarantee the validity of the obtained results. The modified Petrov–Galerkin approach was adopted to correctly handle the convective term. Linear triangular elements were used. The code was flexible enough to treat any polygonal boundary shape with different thermal boundary condition. The number of nodes in the axial direction ranged from 70 to 150, while in the direction normal to the flow it was typically 25. Integration scheme used was Gauss-quadrature with 10 points per dimension (i.e. 100 points in each element). Sparse matrices were constructed and solved using Gauss–Seidel relaxation technique with successive under relaxation. The convergence criterion for a relaxation (j) was based on a combination of absolute and relative error as follows:

$$|\theta^{(j)} - \theta^{(j-1)}| < 10^{-11} + \max\left(|\theta^{(j)}|, |\theta^{(j-1)}|\right) * 10^{-11}. \tag{5.1}$$

The constructed code concentrated on the periodic steady state solution, disregarding the initial transient establishment of oscillations. Since the solution of this problem is periodic, it was expanded in Fourier series with Ω as the fundamental frequency

$$\theta(x,r,t) = \theta_o(x,r) + \mathrm{Re}\left\{\sum_{n=1}^{N} \theta_n(x,r)\mathrm{e}^{\mathrm{i}n\Omega t}\right\}, \tag{5.2}$$

where Re denotes the real part, and i is the imaginary factor and N the retained number of frequencies. Substituting for θ in (2.12) and applying Galerkin technique breaks the transient problem into a set of N coupled complex problems, the nth problem is written as

$$Pr\left(\mathrm{i}n\Omega\theta_n + 2/(1+\delta_{n0})\sum_{m=0}^{N} 1/\tau\int_0^\tau u(t)\theta_m \mathrm{e}^{\mathrm{i}(n+m)\Omega t}\,\mathrm{d}t\right) = \nabla^{2^*}\theta_n. \tag{5.3}$$

Detailed solution for all combination of control parameters showed that the amplitudes of all higher-order harmonics are always negligible compared to that of the fundamental frequency. Hence the summation will be limited to $N = 1$, (i.e. $n, m = 0, 1$).

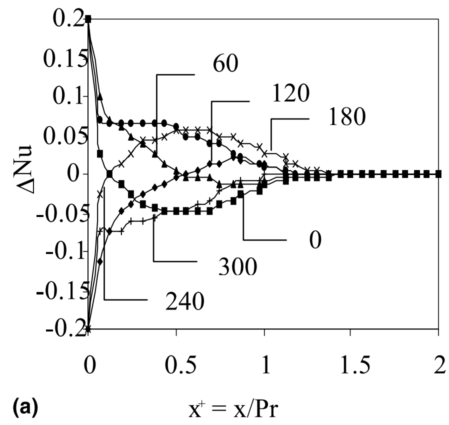
Full validation of the code was obtained through the use of a complete set of test cases each having a known solution. By complete it is meant that each individual feature in the code had a specific problem that tests it. Let us enumerate these features:

1. conduction term;
2. convection term;
3. inertia term;
4. source term (depending on or independent of T);
5. variable thermal conductivity;
6. developing or fully developed;
7. problem domain (Cartesian or cylindrical);
8. boundary condition type (Dirichlet, Neumann or 3rd type).

The test set consisted of the following cases:

1. Conduction in a square with constant heat source and three different types of boundary conditions, which involves features 1, 4 (independent source), 7 (Cartesian) and 8.
2. One-dimensional heat transfer in a fin, which involves features 1 and 4 (dependent source).
3. One-dimensional conduction with variable thermal conductivity, which involves features 1 and 5.
4. Developing and fully developed steady convection in a circular tube with constant wall temperature, which involves features 2, 6, 7 (cylindrical) and 8 (Dirichlet).
5. Developing and fully developed steady convection in a circular tube with constant wall heat flux, which involves features 2, 6, 7 (cylindrical) and 8 (Neumann).
6. Fully developed pulsating flow in a circular cylinder subjected to constant wall heat flux, which involves features 2, 3, 6 (developed), 7 (cylindrical) and 8 (Neumann).
7. Developing and fully developed pulsating flow in a channel subjected to constant wall temperature, which involves features 2, 3, 6, 7 (Cartesian) and 8 (Dirichlet).

Hence every single term in the equation and every single line in the code had a test case to validate the results in situations where the term is non-zero or the line was effectively executed. The first five test cases have analytical solutions using standard techniques. Test case 6 has an analytical solution that was obtained here in Section 3. Test case 7 was an exception since its analytical solution is not known. The instantaneous Nu obtained here was compared in Fig. 4 with that of Kim et al. [4]. It is clear that the results are sufficiently close to validate the code. Differences are mainly due to the difficulty in obtaining precise numbers manually from the figure published in [4]. The error in the first six cases never exceeded 10^{-6} for one-dimensional problems, 10^{-5} for 2D conduction problems and 10^{-3} for 2D convection



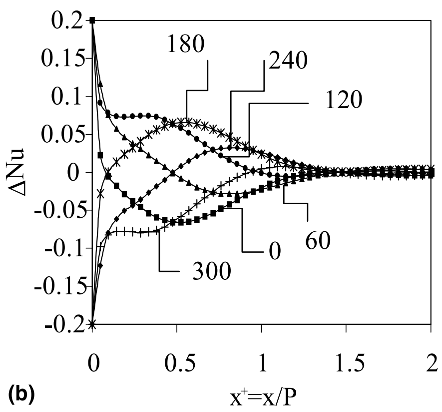


Fig. 4. Results of developing thermal profile for the constant wall temperature test case: (a) results of Kim et al. [4]; (b) present work.

problems. It was also verified that the numerical solution, for the thermally developing region of a circular cylinder in a pulsating flow, tends to the analytical solution for fully developed flow downstream of the duct. Hence, the validity of the code is rigorously established.

6. Finite wall thermal resistance

The effect of each parameter (namely *Re*, Ω, β, *Pr* and *Bi*) will be investigated. Generally speaking, variations in ΔNu were relatively small, which means that pulsation has little effect on the time average Nusselt number for laminar flow.

(1) *Effect of Re*. As mentioned earlier, if we take as a characteristic length in the axial direction $R * Re$, then the role of *Re* will be limited to the axial conduction term. Hence, increasing *Re* would have as a sole effect stretching the development zone proportionally to *Re* (Fig. 5). Note that as x tends to infinity, the oscillations damp out and the curve tends to a small negative number corresponding to the fully developed case.

(2) *Effect of* Ω. As mentioned earlier, ΔNu exhibits a spatial frequency along the tube length, which damps out as we approach fully developed conditions (Fig. 6(a)). If we redraw the results in terms of $x\Omega$ (Fig. 6(b)), we find the following interesting behavior: all curves acquire the same relative wavelength but with different amplitudes. Hence the origin of these oscillations is related to the convection of upstream conditions along the tube length.

To explain the mechanism, consider for simplicity a square wave pulsating flow with a flat velocity profile that starts after a long steady period (Fig. 7(a)), and assume that quasi-static conditions hold during pulsation. Fig. 7(b) gives the slope of temperature rise along the tube corresponding to each constant velocity (high, average and low velocities). At the beginning of oscillations, the temperature distribution corresponded to the median slope. During the first half period, velocity is high; hence each particle will experience a temperature rise along the low slope. At the end of that half period we get a distribution as in Fig. 7(c). At the second half

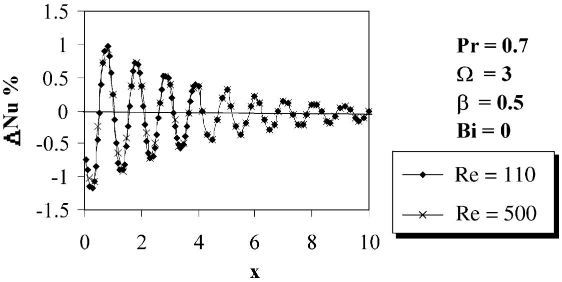


Fig. 5. Effect of *Re*.

period, each particle will start from the previous temperature and will undergo a temperature rise along the high slope, to get a distribution as in Fig. 7(d). It is clear that this will produce a spatial wave having a wavelength proportional to u_o and τ, the oscillation period. Points at different axial locations will undergo temperature oscillations of different amplitudes. They will also have different phase w.r.t. the phase of velocity oscillations. These oscillations damp out as we progress along the duct due to axial mixing. In fact, the actual flow scheme does not correspond to a flat velocity profile. Different layers of the fluid oscillate at different phases, which produces mixing between different layers. The impact of the above scheme on the time average heat transfer coefficient h_o is as follows. The velocity profile has different shapes at accelerating and decelerating half periods. It is relatively steeper near the wall and more flat towards the center at the decelerating half period. Hence, the instantaneous heat transfer coefficient is slightly higher in this half period than in the accelerating one. Thus, the time average h_o will be higher or lower than the steady one, according to the local phase of temperature variation. If at a given axial location high h corresponded to the same period where ΔT is maximum, then h_o will be higher than the steady one h_{st}. At other axial locations where maximum ΔT occurs when h is low, then h_o will be lower than h_{st}.

(3) *Effect of* β. Evidently, increasing the pulsation amplitude increases the difference ΔNu. The relation is parabolic, which is also plausible due to the quadratic interaction between velocity and temperature oscillations in the convective term (Fig. 8).

(4) *Effect of Pr*. As expected, the higher the Prandtl number is, the lower would be the amplitude of oscillations in the thermal field (Fig. 9). In fact, higher *Pr* corresponds to higher C_f and to lower k_f. Increasing heat capacity C_f will decrease temperature swing for the same amount of heat and fluid flow, i.e. will damp out effects due to pulsation. Decreasing k_f would decrease the ability of wall effects to penetrate in the main stream. The combination of both effects results in a decrease in the effect of pulsation on the time average *Nu* as *Pr* increases.

(5) *Effect of Bi*. Simulation results showed that as *Bi* increases pulsation effect increases. The increase is relatively slight in amplitude, but extends to a greater distance along the pipe length, before dying at the fully developed region, as shown in Fig. 10.

To conclude this section, the following remarks will be given about radiation type boundary conditions. Transient radiation has recently received attention due to its numerous industrial applications including laser assisted machining [14] and glass forming [15]. A direct application to the configuration studied here, would be that of an exhaust tube where flue gases exhibit a pulsated flow inside the tube, while the outside tube surface

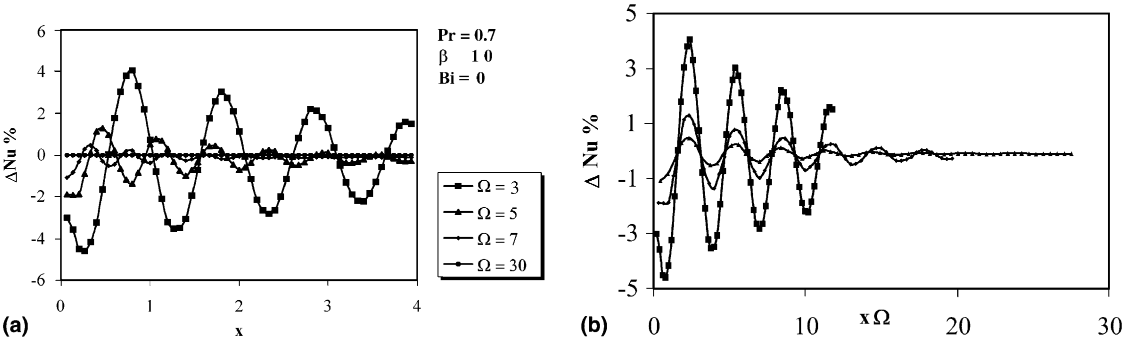


Fig. 6. Effect of pulsation frequency on heat transfer: (a) dependence of Nu on x; (b) dependence of Nu on $x\Omega$.

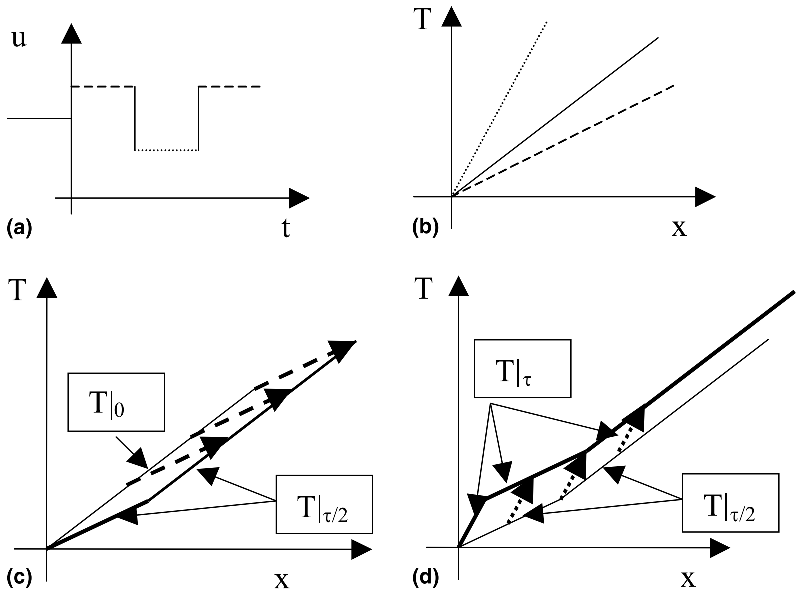


Fig. 7. Mechanism of creation of spatial oscillations: (a) time variation of velocity; (b) slope of spatial variation of T; (c) situation after the first half cycle; (d) situation after one cycle.

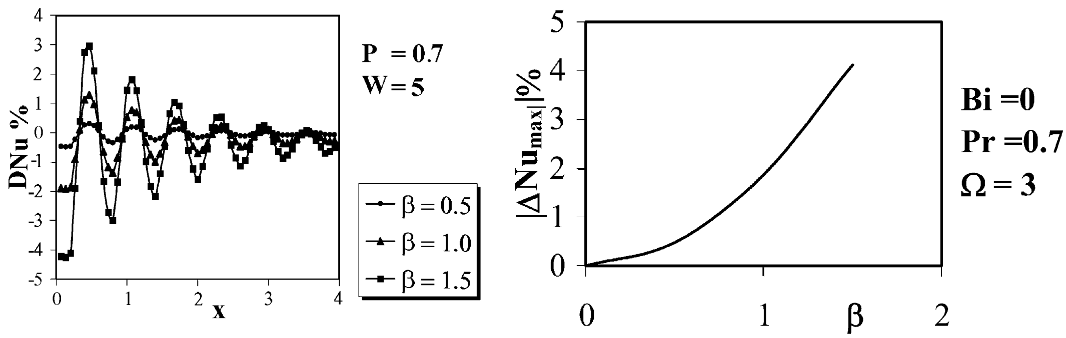


Fig. 8. Effect of β on heat transfer.

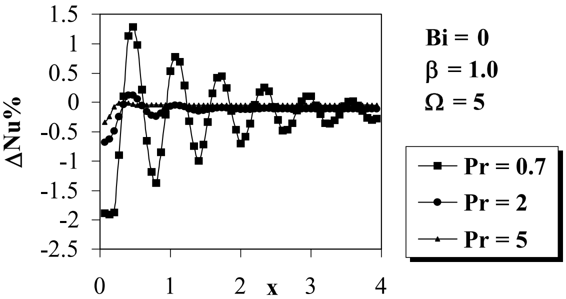


Fig. 9. Effect of Pr on heat transfer.

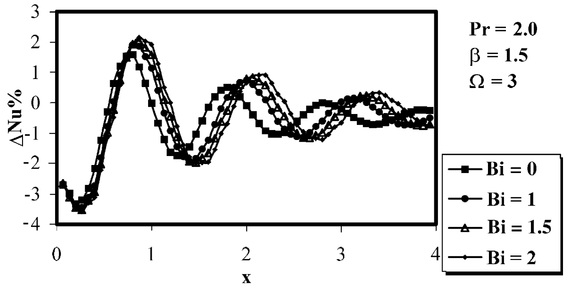


Fig. 10. Effect of Bi number on heat transfer.

may have a sufficiently high temperature such that radiation cannot be neglected. Under these conditions, the problem becomes non-linear, which gives rise to the following interesting fact: The time average heat transfer rate is greater than the rate due to the time average temperature difference. To be more specific, if the dimensionless temperature had an expression of the form (5.2) with $N = 1$ for simplicity, then the time average heat transfer rate between tube wall and surroundings q_{wo} would be

$$q_{\mathrm{wo}} = \sigma(\Delta T_{\mathrm{ref}})^4\left[\theta_{\mathrm{o}}^4 - \theta_{\mathrm{amb}}^4 + 3\theta_0^2\theta_1^2 + \theta_1^4/4\right],$$

while the rate of heat transfer due to the average temperature difference q'' would be

$$q'' = \sigma(\Delta T_{\mathrm{ref}})^4[\theta_{\mathrm{o}}^4 - \theta_{\mathrm{amb}}^4],$$

which is certainly less than q_{wo}. The same could be said about natural convection with the surroundings. In both cases, non-linearity combined with pulsation may result in a noticeable enhancement of the time average heat transfer rate. These subjects are currently under investigation.

7. Finite wall thermal capacitance

Wall thermal inertia may considerably modify system dynamics. These effects were studied [16] in the context of dynamic characterization of flux gages, as well as that of measuring physical properties [17]. Both have considered the transient conduction equation in the solid wall. As a first approximation, wall temperature will be assumed here constant along the wall thickness s, but variable along the pipe axis. This assumption is valid for small thickness, highly conductive walls. It allows us to simplify the problem, by modeling the effect by a modification in boundary conditions. If this effect proves to be important, more detailed analysis would be justified. Hence the new wall boundary condition would be in the form

$$k_{\mathrm{f}}\,\partial T/\partial r'\big|_{r'=R} = -\rho_{\mathrm{w}} s C_{\mathrm{w}} \partial T/\partial t'\big|_{r'=R} + h_{\mathrm{out}}\left(T_{\mathrm{amb}} - T|_{r=R}\right) + q_{\mathrm{w}}. \tag{7.1}$$

In dimensionless form, this yields

$$\partial\theta/\partial r|_{r=1} = 1 + Bi(1-\theta) - Pr\,\mathrm{Ir}\partial\theta/\partial t|_{r=1},$$

where $\mathrm{Ir} = s/R\ \rho_{\mathrm{s}}/\rho_{\mathrm{f}}\ C_{\mathrm{s}}/C_{\mathrm{f}}$ is simply the ratio of wall thermal inertia to that of the fluid. The added time derivative term was tested against a simple transient conduction problem, having a known analytical solution.

Simulation results (Fig. 11) show that wall thermal inertia can quickly damp out pulsation effects, especially at high frequencies. Not only does the amplitude of oscillation decrease, but also the length necessary to establish the fully developed case sharply shrinks to only few spatial periods.

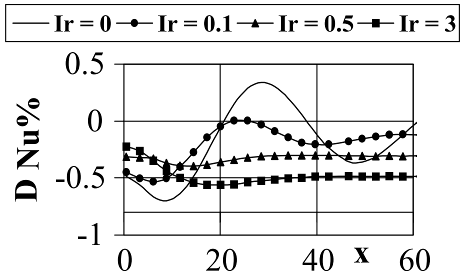


Fig. 11. Effect of wall thermal inertia.

8. Conclusions

Heat transfer in laminar incompressible pulsating flow in a duct has been theoretically analyzed. The thermally fully developed case was solved analytically, while the thermally developing case was calculated using FEM. Numerical results were fully validated through the use of a complete set of test cases having known analytical or numerical solutions. The main contributions of this work are the following:

1. An analytical solution of the thermally fully developed case with constant wall heat flux was redone

in order to correct an error in a previously published work [6]. The error resulted in an unphysical behavior at high frequency, which was avoided here. The corrected solution eliminates confusion about the role of pulsation in the fully developed region, by proving that its effect is negligible (<1%).
2. A general and rational definition of time average Nusselt number was constructed. It applies to different boundary conditions, guarantees compliance with energy balance and enables the use of the obtained thermal resistance in series or parallel with other thermal resistances in the system, which is not the case of previously proposed averages.
3. The thermally developing region showed relatively greater sensitivity to pulsation than did the fully developed region. Differences up to about 6% between the time average Nusselt number in the thermally developing region and that of the corresponding steady problem (ΔNu) were observed. The difference ΔNu varied sinusoidally along the tube length with a spatial period, which was equal to the average fluid velocity multiplied by the time period, and diminishing amplitude according to a mechanism that has been explained in this work. The difference ΔNu changes sign along tube length, but its space average is negative.
4. The effects of different control parameters were analyzed. The sole effect of Re number was to stretch the thermally developing region. Pulsating effect on heat transfer increases with pulsation amplitude β but decreases with pulsation frequency Ω as well as Pr number. Different kinds of boundary conditions were considered. A Biot number Bi was defined to characterize a finite wall thermal resistance. As Biot increases from zero to infinity, pulsation effect on the time average Nusselt number slightly increases. A finite wall thermal inertia was also considered, characterized by Ir the ratio of wall to fluid thermal inertia. It was found that wall thermal inertia damps out pulsation effect.
5. The controversial subject whether pulsation enhances or degrades heat transfer is answered here as follows: As long as we restrict ourselves to laminar incompressible flow, with linear boundary conditions, the effect of pulsation on the time average heat transfer coefficient tends to be negative, but remains relatively small. Non-linear boundary conditions, (e.g. radiation and natural convection) combined with pulsation may result in a noticeable enhancement of the time average Nusselt number, as shown by a quick discussion in this work. Of course turbulence, which is also a non-linear phenomenon, would give rise to heat transfer enhancement as a result of pulsation. Hence, theoretical and experimental studies in pulsating flow should concentrate on non-linear phenomena.

Appendix A

Moschandreou and Zamir [6] have attempted to obtain an analytical solution of the fully developed case. Eqs. (20) and (21) in their text are rewritten here in their notations:

$$h''(r) + h'(r)/r + \alpha^2 h(r) = g(r), \tag{A.1}$$

$$g(r) = (-\mathrm{i}/\omega)[1 - J_0(\lambda r)/J_0(\lambda)], \tag{A.2}$$

where

$$\alpha^2 = -\mathrm{i}\omega, \quad \lambda = \alpha/\sqrt{Pr}, \tag{A.3}$$

which are equivalent to (3.5a), (2.11b) and (3.5b), respectively. They have obtained the following solution:

$$h(r) = (\pi/2)\alpha\{J_0(\alpha r)[-c_1 I_1 + I_2(r)] + Y_0(\alpha r) I_3(r)\}, \tag{A.4}$$

where

$$c_1 = \mathrm{Re}\big((\partial Y_0(\alpha r)/\partial r)|_{r=1}/(\partial J_0(\alpha r)/\partial r)|_{r=1}\big), \tag{A.5}$$

$$I_1 = \int_0^1 J_0(\alpha\tau) g(\tau)\, \mathrm{d}\tau, \tag{A.6}$$

$$I_2(r) = \int_r^1 Y_0(\alpha\tau) g(\tau)\, \mathrm{d}\tau, \tag{A.7}$$

$$I_3(r) = \int_0^r J_0(\alpha\tau) g(\tau)\, \mathrm{d}\tau. \tag{A.8}$$

This solution does not satisfy (A.1) as will be shown here. In fact, the first derivative of h is

$$h'(r) = (\pi/2)\alpha\{\alpha J_0'(\alpha r)[-c_1 I_1 + I_2(r)] + \alpha Y_0'(\alpha r) I_3(r) + Z_1(r)\},$$

where

$$Z_1(r) = [J_0(\alpha r) I_2'(r) + Y_0(\alpha r) I_3'(r)].$$

The function $Z_1(r)$ vanishes by virtue of (A.7) and (A.8) to leave

$$h'(r) = (\pi/2)\alpha^2\{J_0'(\alpha r)[-c_1 I_1 + I_2(r)] + Y_0'(\alpha r) I_3(r)\}. \tag{A.9}$$

The second derivative of h is

$$h''(r) = (\pi/2)\alpha^2\{\alpha J_0''(\alpha r)[-c_1 I_1 + I_2(r)] + \alpha Y_0''(\alpha r) I_3(r) + Z_2(r)\}, \tag{A.10}$$

where

$$\begin{aligned} Z_2(r) &= J_0'(\alpha r) I_2'(r) + Y_0'(\alpha r) I_3'(r) \\ &= [-J_0'(\alpha r) Y_0(\alpha r) + Y_0'(\alpha r) J_0(\alpha r)] g(r) \\ &= [J_1(\alpha r) Y_0(\alpha r) - Y_1(\alpha r) J_0(\alpha r)] g(r) \\ &= [2/(\pi\alpha r)] g(r). \end{aligned} \tag{A.11}$$

(The last transformation was based on the Wronskian of Bessel functions.)

Hence by substituting (A.4), (A.9) and (A.10) in (A.1) we get for the LHS using (A.11):

$$\begin{aligned}\mathrm{LHS} = {} & (\pi/2)\alpha\{\alpha^2 J_0''(\alpha r) + \alpha J_0'(\alpha r)/r \\ & + \alpha^2 J_0(\alpha r)\}[-c_1 I_1 + I_2(r)] \\ & + (\pi/2)\alpha\{\alpha^2 Y_0''(\alpha r) + \alpha Y_0'(\alpha r)/r \\ & + \alpha^2 Y_0(\alpha r)\} I_3(r) + \alpha g(r)/r. \qquad \text{(A.12)}\end{aligned}$$

The terms between curly brackets vanish since both J_0 and Y_0 satisfy Bessel equation, hence

$$\mathrm{LHS} = \alpha g(r)/r,$$

which is not equal to the RHS of (A.1).

Let us perform the same steps on the solution obtained in this work, given by (3.7), (3.9a), (3.9b) and (3.10), that is rewritten here for reference

$$f(r) = J_0(Kr)[I_4(r) + I_5(r)] + Y_0(Kr) I_6(r), \qquad \text{(A.13)}$$

where

$$I_4(r) = \int_0^r \left[-\pi J_0(K\tau) Y_0'(K)/2J_0'(K) \right] \tau g(\tau)\, \mathrm{d}\tau, \qquad \text{(A.14)}$$

$$\begin{aligned} I_5(r) = \int_r^1 & \pi\left[\left(Y_0(K\tau) J_0'(K) \right.\right. \\ & \left.\left. - Y_0'(K) J_0(K\tau) \right) / 2J_0'(K) \right] \tau g(\tau)\, \mathrm{d}\tau, \qquad \text{(A.15)}\end{aligned}$$

$$I_6(r) = \int_0^r \pi [J_0(K\tau)/2] \tau g(\tau)\, \mathrm{d}\tau. \qquad \text{(A.16)}$$

The first derivative of f is

$$f'(r) = KJ_0'(Kr)[I_4(r) + I_5(r)] + KY_0'(Kr) I_6(r) + Z_3(r),$$

where

$$Z_3(r) = J_0(Kr)\left[I_4'(r) + I_5'(r) \right] + Y_0(Kr) I_6'(r).$$

By direct substitution of integrals from (A.14)–(A.16) into the definition of Z_3, it is clear that it vanishes leaving

$$f'(r) = KJ_0'(Kr)[I_4(r) + I_5(r)] + KY_0'(Kr) I_6(r). \qquad \text{(A.17)}$$

The second derivative of f is

$$f''(r) = K^2 J_0''(Kr)[I_4(r) + I_5(r)] + K^2 Y_0''(Kr) I_6(r) + Z_4(r),$$

where

$$Z_4(r) = KJ_0'(Kr)\left[I_4'(r) + I_5'(r) \right] + KY_0'(Kr) I_6'(r).$$

By substitution of the integrals (A.14)–(A.16) and using the Wronskian as before, we get

$$Z_4 = g(r).$$

Finally, by substituting $f,\ f',\ f''$ in (3.5a) it is clear that the solution obtained here satisfies it.

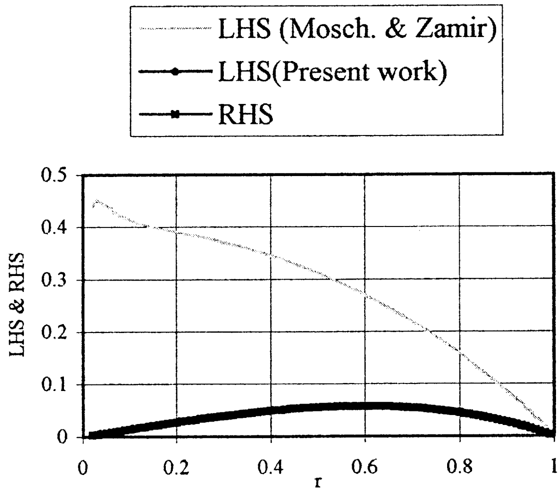


Fig. 12. Comparing the solutions of Moschandreou and Zamir and the present work.

To obtain a second confirmation of the above calculations, all derivatives were numerically calculated for both solutions and directly substituted in the differential equation. Eq. (A.1) was multiplied by r in order to be compared with (3.5a). A value of $Pr = 1$ was also taken to eliminate differences due to different time scales used. Results are drawn in Fig. 12 clearly showing that the solution of Moschandreou and Zamir [6] was incorrect, while that obtained here was correct.

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